

Possibility of a Light Pulse with Speed Greater than c

Xian-jian Zhou

Institute of High Energy Physics, Academia Sinica

P.O.Box 918-4, Beijing 100039

People's Republic of China

Abstract

In two models it is shown that a light pulse propagates from a vacuum into certain media with velocity greater than that of a light in a vacuum (c). By numerical calculation the propagating properties of such a light are given.

Recently L. J. Wang and his collaborators in their experiment [1] have found that the group velocity of a laser pulse in particularly prepared atomic caesium gas can much exceed that of light in a vacuum (c). Here we will explore the problem from a theoretical point of view. Many years ago A. Sommerfeld rigorously proved [2] that the velocity of light pulse can not exceed c in absorption media. We call this conclusion as basic theorem afterwardz. In the following we will first examine why usually the basic theorem holds and in what condition it will be violated. Then two models are proposed, where with precise numerical calculation we will show that the basic theorem is indeed violated. Therefore the properties of light propagating in media of the models are given. We believe that these properties may appears in more realistic models.

Let a light pulse propagating along x axis toward its positive direction in a vacuum ($x < 0$), at time $t = 0$ arriving at $x = 0$, and then entering into a medium ($x \geq 0$) afterwards. At $x = 0$, the amplitude of the light pulse changes with time t as

$$f(t) = \begin{cases} F(t) & t \geq 0 \\ 0 & t < 0 \end{cases} \quad (1)$$

Usually $f(t)$ can be rigorously expressed in Fourier integration as

$$f(t) = \text{Re} \int_0^\infty A(n) e^{-int} dn \quad (2)$$

For simplicity suppose there is no reflection of light at $x = 0$. The amplitude of the light pulse entering into the medium ($x \geq 0$) is [2]

$$f(t, x) = \text{Re} \int_0^\infty A(n) e^{-int+ikx} dn \quad (3)$$

where $k = n\mu(n)/c$ and $\mu(n)$ is the complex refractive index of the medium, which depends on the frequency of incident light (dispersion). In vacuum ($x < 0$), the amplitude of the light pulse is

$$\begin{aligned} g(t, x) &= \text{Re} \int_0^\infty A(n) e^{-int+inx/c} dn \quad (x < 0) \\ &= f(t - x/c) = \begin{cases} F(t - x/c) & t - x/c \geq 0 \\ 0 & t - x/c < 0 \end{cases} \end{aligned} \quad (4)$$

The shape of the light pulse propagating in a vacuum does not change because all its Fourier components in (4) propagate with the same velocity c and do not decay. In particular, all these components completely cancel each other in the space-time region $t - x/c < 0$ (or $\theta = t c/x < 1$) as long as the light pulse propagates in a vacuum. One may think that such a cancellation may not occur when light pulse propagates in a medium because of dispersion and absorption. But Sommerfeld proved that such cancellation also occurs when light pulse propagates in absorption media, which is just the basic theorem. In his proving Sommerfeld let

$$f(t) = \begin{cases} \sin \nu t & t \geq 0 \\ 0 & t < 0 \end{cases} \quad (5)$$

so that (3) becomes [2]

$$f(t, x) = \frac{1}{2\pi} \text{Re} \int_c e^{-i(nt-n\mu(n)x/c)}_{/(n-\nu)} dn \quad (6)$$

where $\mu(n)$ is taken as

$$\mu^2 = 1 + \frac{a^2}{n_0^2 - 2i\rho n - n^2} \quad , \quad (7)$$

which is the Lorentz-Lorenz refraction formula. a^2, n_0, ρ are the constants of the medium. n_0, ρ represent the characteristic absorption frequency, damping constant of the medium. Usually $\rho > 0$, light propagating in the medium decays. The integration path c in (6) is shown in Fig.1, which is along the real axis of n from $+\infty$ to $-\infty$ through $n = \nu$ by a small semicircle in the upper half of the complex plane. $\mu(n)$ in (7) has branch points:

$$\begin{aligned} U_{1,2} &= -i\rho \pm \sqrt{n_0^2 - \rho^2} \quad , \text{ where } \mu = \infty \quad ; \\ N_{1,2} &= -i\rho \pm \sqrt{n_0^2 + a^2 - \rho^2} \quad , \text{ where } \mu = 0 \quad . \end{aligned}$$

We joining U_1 to N_1 and U_2 to N_2 by two branch lines, which lie in the lower half of the complex plane, when $\rho > 0$. Because there is no singularity and branch lines of the integrand of (6) in the upper half plane, one can replace the integration path c by u in Fig.1, which is parallel to the real axis in the upper half plane. When u moves to infinity in the upper half plane, $\mu \rightarrow 1$ and when $t - x/c < 0$, then (6) = 0, which is just the basic theorem. But when $\rho < 0$ (this is our model 1), the branch lines U_1N_1 and U_2N_2 lie in the upper half plane and the integration path u is not equivalent to the path c in (6). Now the equivalent path u should be taken as $u_1 + u_2 + u_3 + u_4$ in Fig.1. By the same argument above, integration along u_1 vanishes. Integrations along a pair of u_2 cancel each other. The remaining integrations along u_3 and u_4 usually do not vanish because the branch lines lie in them. Therefore when $\rho < 0$ (model 1), the basic theorem does not hold. In the following we will do numerical integration of (6) to show that the basic theorem is indeed violated and to see what happens. $\rho < 0$ means propagation of light in the medium is gain-assisted light propagation.

From (6) we get

$$f(t, x) = \frac{1}{2} \text{Re}[ie^{\gamma\omega(\bar{\nu})}] + \frac{1}{2\pi} \text{Re} \int_0^\infty [e^{\gamma w(-z+\bar{\nu})} - e^{\gamma w(z+\bar{\nu})}] \frac{dz}{z} \quad , \quad (8)$$

where $\gamma w(\bar{n}) = -int + in\mu x/c$; $\gamma = xn_0/c$, $\bar{n} = n/n_0$, $\bar{\nu} = \nu/n_0$, $\bar{a}^2 = a^2/n_0^2$, $\bar{\rho} = \rho/n_0$, all of them are dimensionless. Let

$$w(z) = X(z) + iY(z) \quad . \quad (9)$$

When $z \gg 1$,

$$X(z) \sim -\bar{a}^2 \bar{\rho} / z^2, \quad Y(z) \sim z(1 - \theta) - \bar{a}^2 / (2z), \quad (10)$$

where $\theta = ct/x$.

In reference [2], the typical values of parameters are given as

$$n_0 = 4 \times 10^{16} \text{ s}^{-1}, \quad a^2 = 1.24 \text{ } n_0^2, \quad \rho = 0.07 \text{ } n_0, \quad (11)$$

where the medium is solid or liquid. For gas a^2 is about 1.001.

The numerical integration for (8) is difficult when $|\gamma Y(z)|$ becomes very large and hence $e^{\gamma Y(z)}$ is a very fast oscillatory function of z . In fact when $n_0 = 4 \times 10^{16} \text{ s}^{-1}$ and $x = 1 \text{ cm}$, $\gamma = \frac{4}{3} \times 10^6$ is very large. But if $\gamma < 100$ for example, i.e., $x < 7.5 \times 10^{-5} \text{ cm}$, we can do numerical integration of (8) with high precision. Now the integration of (8) is divided into sum of \int_0^R and \int_R^∞ . With $R \sim 100$ for example, fast oscillatory integrand appears in \int_R^∞ , when $z \gg 1$. Using (10)

$$Re(e^{\gamma \omega(z)}) = e^{-\gamma \bar{a}^2 \bar{\rho} / z^2} \left\{ \cos[\gamma z(1 - \theta)] \cos \frac{\gamma \bar{a}^2}{2z} + \sin[\gamma z(1 - \theta)] \sin \frac{\gamma \bar{a}^2}{2z} \right\},$$

where the fast oscillatory factors are separated in forms of sin and cos functions. Mathematica can do that kind of integration with high precision. As an example, let $\bar{a}^2 = 1.24$, $\bar{\rho} = 0.07$, $\bar{\nu} = 10$, $\gamma = 1$, $\theta = 0.98$ in (8), one may get $f(t, x) = -1.02057 \times 10^{-13}$, which should vanish exactly due to the basic theorem ($\rho > 0, \theta < 1$). Considering the amplitude of incident light pulse is 1, 10^{-13} is a very high accuracy of calculation. Now let $\bar{\rho} = -0.07$ and other parameters unchanged, $f(t, x)$ in (8) is 0.0630255, which is a definite evidence to show that the basic theorem is violated indeed when $\rho < 0$. Because $\gamma = xn_0/c$, $\theta = tc/x$, $f(t, x)$ may be looked as a function of γ and θ , $h(\gamma, \theta)$. Let us fix $\gamma = 1$, i.e., $x = 0.75 \times 10^{-6} \text{ cm}$, and θ change from (-7) to 4, i.e., the time t from $-7/n_0$ to $4/n_0$, the amplitude of light pulse $h(1, \theta)$ in the medium is shown in Fig.2, where $\bar{a}^2 = 1.24$, $\bar{\rho} = -0.07$, $\gamma = 1$, $\bar{\nu} = 10$ ($\nu = 10 n_0$). Again when $\rho < 0$, $h(1, \theta)$ does not vanish and the basic theorem is violated. In Fig.2 even if $t < 0$, the amplitude $h(1, \theta)$ still does not vanish, which means that before the incident light pulse arrives at the medium, the light in the medium is already produced. For convenience, we call the light produced in the medium when $\theta < 1$ as fastlight and that when $\theta > 1$ as normal light. When the basic theorem

holds, the fastlight vanishes. Some maximal values of light amplitude near $\theta = 1$ and their corresponding θ values are listed in table 1, where two characteristics are shown:

(1) The amplitude of fastlight is oscillatory and decays as $\theta \rightarrow -\infty$;

(2) The period for each oscillation for θ less and near 1 are roughly around $\theta = 6$, which means its frequency is near the characteristic frequency n_0 of the medium (the corresponding period is $\theta = 2\pi$). This near equality is due to the fact that the Fourier components of the light pulse having frequencies equal to or near n_0 are most gain-assisted when $\rho < 0$. When $\theta > 1$, the normal light soon oscillates with the frequency ν of the incident light pulse (corresponding period $\theta = 0.2\pi$) and its amplitude is a little bit larger than $1(1.00089)$. When $\bar{\nu} = 1$ with other parameters $\bar{a}^2 = 1.24, \bar{\rho} = -0.07, \gamma = 1$ unchanged the two properties (1) and (2) remain, but the amplitude of the fastlight increases to 6.14 near $\theta = 1$. The amplitude of normal light increases to 7.31 with frequency $\nu = n_0$.

Model 2: the ρ in (7) depends on frequency n as

$$\rho(n) = \begin{cases} \rho_1 & \ell - b < n < \ell + b \\ \rho_2 & n \leq \ell - b \text{ or } n \geq \ell + b \end{cases}, \quad (12)$$

which is not continuous and therefore $\mu(n)$ and the integrand in (6) are not the analytic functions of n . We still can use numerical calculation to see whether the basic theorem holds:

(a) $\bar{a}^2 = 1.24, \rho_1 = -0.07, \rho_2 = 0.07, \ell = 1, b = 0.01, \bar{\nu} = 1, \gamma = 1$. Now light amplitude in the medium are gain-assisted when $(1-b)n_0 < n < (1+b)n_0$ and decays otherwise. Fastlight with above two properties appears again and its amplitude near $\theta = 1$ is 3.81, while that of normal light is about 4.5;

(b) $\rho_1 = 0.02$ with other parameters unchanged as in (a). Now light in the whole frequency range in the medium decays. Still fastlight remains with the two properties, but its amplitude becomes small (0.0757) near $\theta = 1$.

So we may conclude that if $\mu(n)$ in (6) is an analytic function of n with singularities (such as poles or branch lines) appearing in the upper half plane of n , or $\mu(n)$ is not an analytic function of n at all, the basic theorem in general may not hold and fastlight appears. The Fourier components of a light pulse now will not cancel each other in the medium in the space-time region $t - x/c < 0$. This is why fastlight appears.

How to measure the velocity of the light pulse when fastlight appears in the medium? Suppose a light pulse produced in a source at $t = 0$, propagating a distance ℓ_1 in a vacuum, then going through a medium with thickness ℓ_2 . Just after the medium a light pulse detector is put. Usually the fastlight appears in the medium after the light pulse is produced and the amplitude of the fastlight is inceasing when the light pulse is approaching to the medium. If the amplitude of fastlight is able to become large enough to trigger the detector at $t = t_1$, one may take $v = (\ell_1 + \ell_2)/t_1$ as the velocity of the light pulse propagating from the source to the detector, which certainly exceeds c .

Although the models proposed above are not completely realistic, we believe that production of the fastlight and some its properties in the models may remain in a more realistic model, which is under investigation now.

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References

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**Table 1. Some maximal values of light amplitude $h(1, \theta)$ near $\theta = 1$
($\bar{a}^2 = 1.24, \bar{\rho} = -0.07, \gamma = 1$ and $\bar{\nu} = 10$ in model 1)**

θ	-46.86	-40.70	-34.57	-28.44	-22.35	-16.29	-10.27
$h(1, \theta)$	0.00909	0.0125	0.0172	0.0232	0.0316	0.0413	0.0580

-4.59	0.8714	1.1508	1.77912	2.40744	3.03575
0.0742	0.0635	1.0089	1.0089	1.0089	1.0089

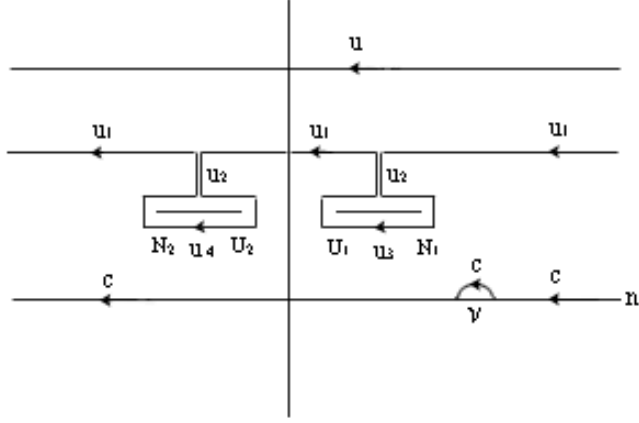


Figure 1: The integration paths of (6).

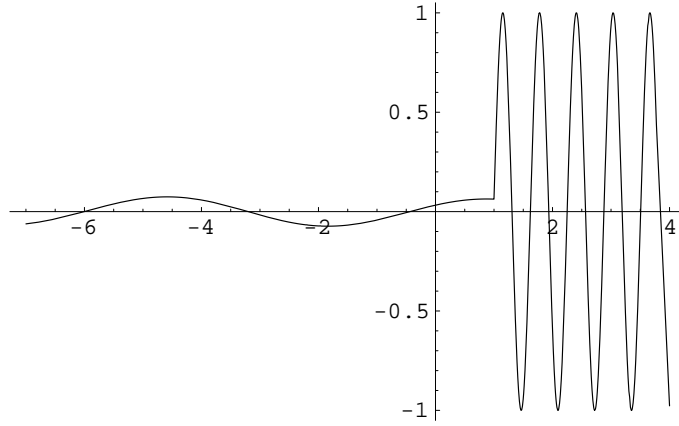


Figure 2: Figure 2. $h(1, \theta)$ ($\bar{a}^2 = 1.24$, $\bar{\rho} = -0.07$, $\gamma = 1$ and $\bar{\nu} = 10$ in model 1.)